TRANSFORMATIONS

Gursharan Singh Tatla

professorgstatla@gmail.com

Translation

$$x' = x + t_x$$
$$y' = y + t_y$$

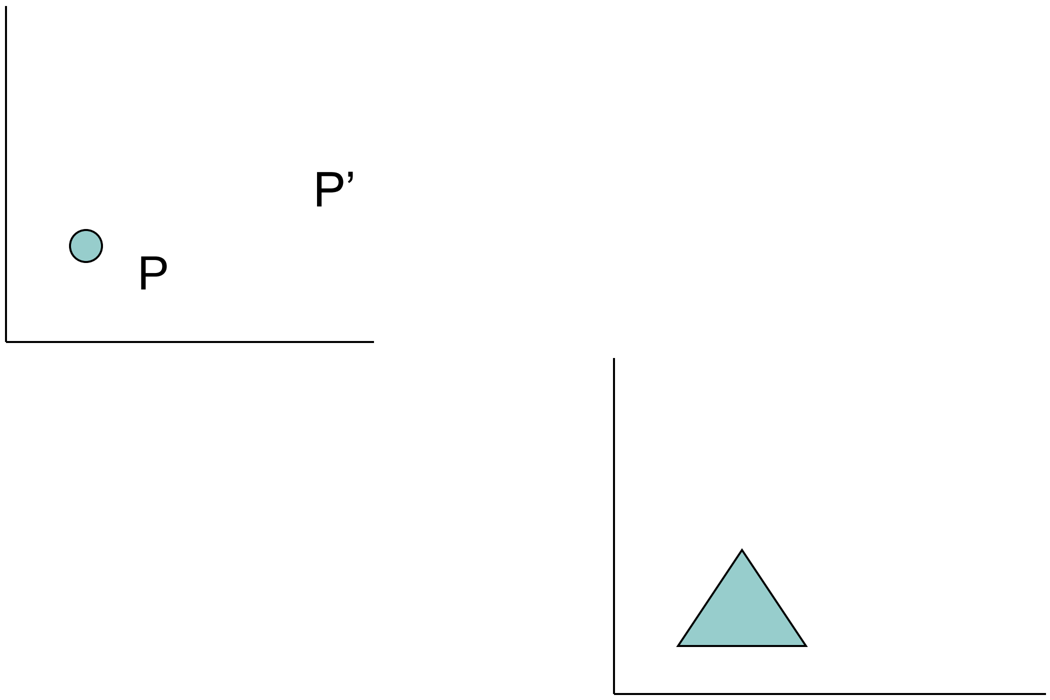
The translation distance pair (t_x, t_y) is called a *translation vector* or *shift vector*

$$P = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad P' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

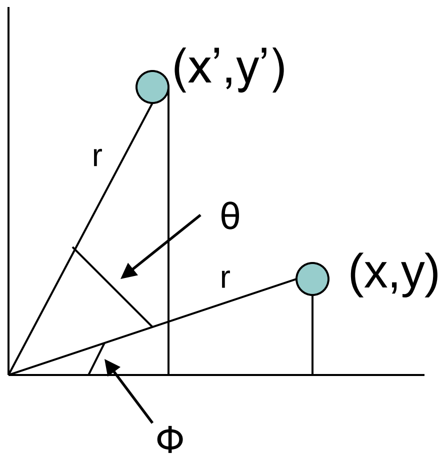
This allows us to write the two dimensional translation equations in the matrix form

$$\mathbf{P' = P + T}$$

Translation Illustration



Rotation 1



The original coordinates of the point in Polar Coordinates are

$X = r\cos(\Phi)$ $\quad$ $y = r\sin(\Phi)$

Rotation 2

- $x' = r\cos(\Phi + \theta) = r\cos\Phi\cos\theta - r\sin\Phi\sin\theta$

  $Y' = r\sin(\Phi + \theta) = r\cos\Phi\sin\theta + r\sin\Phi\cos\theta$

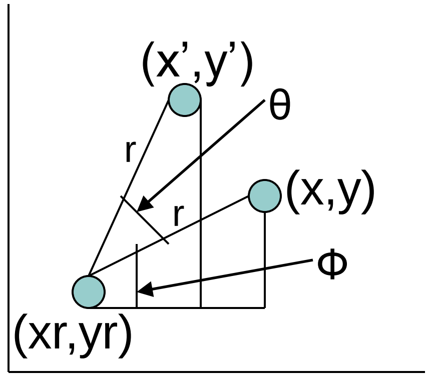
- $x' = x\cos\theta - y\sin\theta$

  $Y' = x\sin\theta + y\cos\theta$

- P' = R.P

- $$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation 3



- $x' = xr + (x - xr)\cos\theta - (y - yr)\sin\theta$
- $Y' = yr + (x - xr)\sin\theta + (y - yr)\cos\theta$

Scaling 1

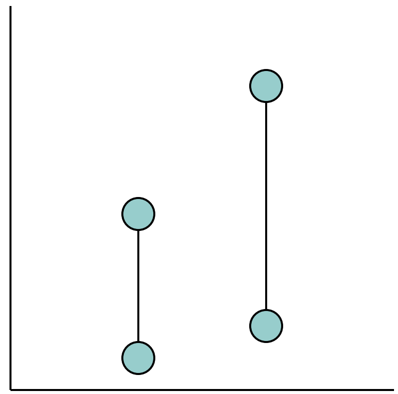
- $X' = x.\,sx \quad y' = y.sy$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} \quad \begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix}$$

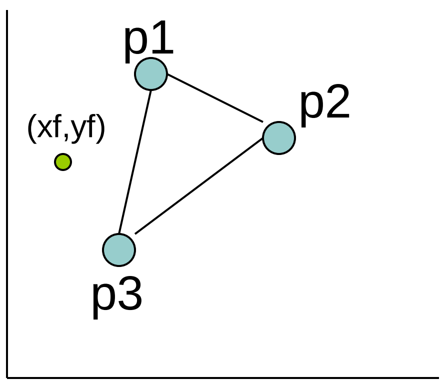
- $P' = S\,.\,P$

Turning a square into a rectangle with scaling factors sx= 2 and sy =1.5

Scaling 2



Using sx= sy = 0.5 the line is reduced in size and moved closer to the origin



Scaling relative to a chosen fixed point (xf, yf) . Distances from each polygon vertex to the fixed point are scaled y transformation equations

X’ = xf + (x- xf) sx

Y’ = yf + (y-yf) sy

Transformations as Matrix operations

$$\begin{bmatrix} X' \\ Y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$P' = T(tx, ty) . P$$

Translations

Rotations

$$\begin{bmatrix} X' \\ Y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$P' = R(\theta) \, . P$$

Scaling

$$\begin{bmatrix} X' \\ Y' \\ 1 \end{bmatrix} = \begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$P' = S(sx, sy) \, .P$$

Successive translations

$$\begin{bmatrix} 1 & 0 & tx1+tx2 \\ 0 & 1 & ty1+ty2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx1 \\ 0 & 1 & ty1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & tx2 \\ 0 & 1 & ty2 \\ 0 & 0 & 1 \end{bmatrix}$$

Successive translations are additive

$$P' = T(tx1, ty1) .[T(tx2, ty2)]\ P$$
$$= \{T(tx1, ty1). T(tx2, ty2)\}.P$$

$$T(tx1, ty1). T(tx2, ty2) = T(tx1+tx2 , ty1 + ty2)$$

Successive rotations

- By multiplying two rotation matrices , we can verify that two successive rotations are additive

$$P' = R(\theta 2) . \{ R(\theta 1). P \}$$
$$= \{ R(\theta 2). R(\theta 1)\}.P$$

$$\{ R(\theta 2). R(\theta 1)\} = R(\theta 1 + \theta 2)$$

$$P' = R(\theta 1 + \theta 2) . P$$

Successive Scaling operations

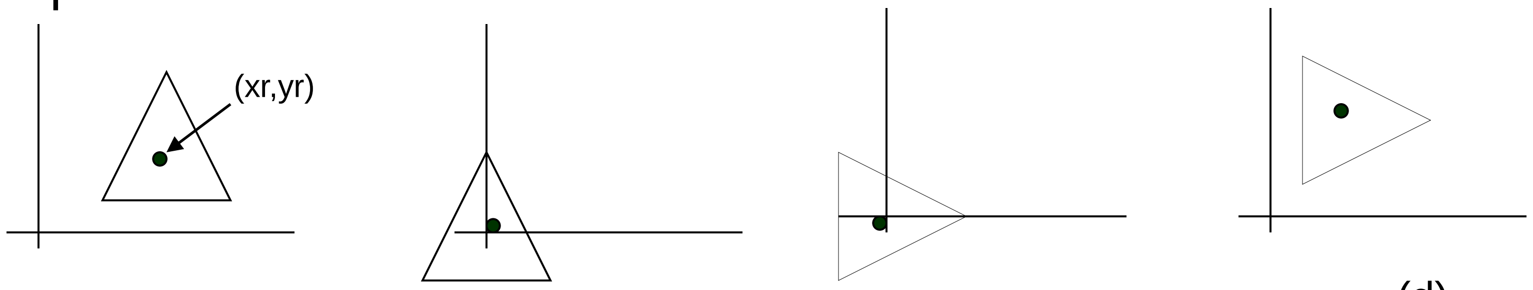
$$\begin{bmatrix} Sx1.sx2 & 0 & 0 \\ 0 & sy1sy2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} sx1 & 0 & 0 \\ 0 & sy1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx2 & 0 & 0 \\ 0 & sy2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

S(sx2,sy2).S(sx1,sy1) = S(sx1.sx2 , sy1,sy2)

The resulting matrix in this case indicates that successive scaling operations are multiplicative

General pivot point rotation

- Translate the object so that pivot-position is moved to the coordinate origin
- Rotate the object about the coordinate origin
- Translate the object so that the pivot point is returned to its original position



(a)

Original Position of Object and pivot point

(b)

Translation of object so that pivot point (xr,yr)is at origin

(c)

Rotation was about origin

(d)

Translation of the object so that the pivot point is returned to position (xr,yr)

General pivot point rotation

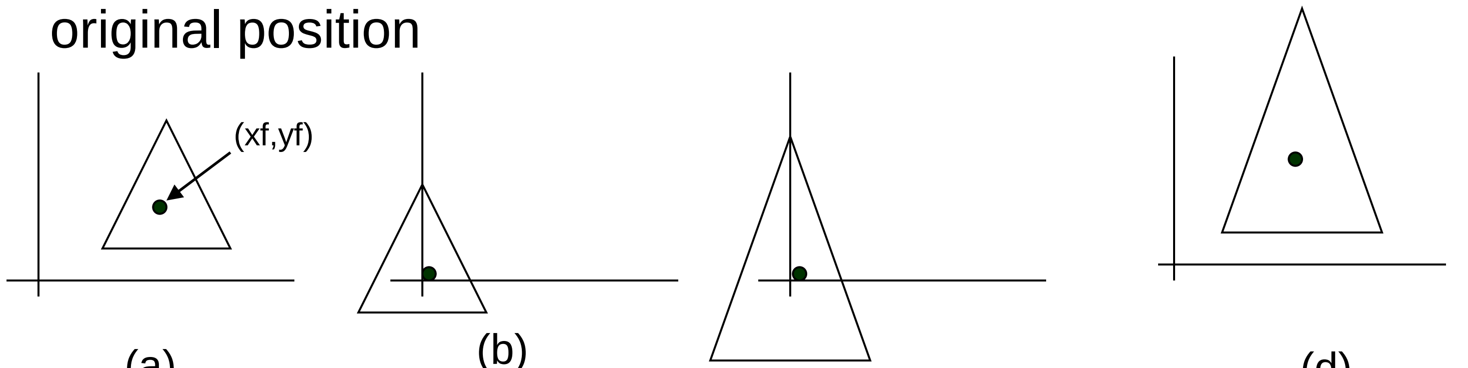
$$\begin{bmatrix} 1 & 0 & xr \\ 0 & 1 & yr \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -xr \\ 0 & 1 & -yr \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & xr(1-\cos\theta) + yr\sin\theta \\ \sin\theta & \cos\theta & yr(1-\cos\theta) - xr\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

Can also be expressed as T(xr,yr).R(θ).T(-xr,-yr) = R(xr,yr,θ)

General fixed point scaling

- Translate object so that the fixed point coincides with the coordinate origin
- Scale the object with respect to the coordinate origin
- Use the inverse translation of step 1 to return the object to its original position



(a) Original Position of Object and Fixed point

(b) Translation of object so that fixed point (xf,yf)is at origin

(c) scaling was about origin

(d) Translation of the object so that the Fixed point is returned to position (xf,yf)

General pivot point Scaling

$$\begin{bmatrix} 1 & 0 & xf \\ 0 & 1 & yf \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -xf \\ 0 & 1 & -yf \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} sx & 0 & xf(1- sx) \\ 0 & sy & yf(1- sy) \\ 0 & 0 & 1 \end{bmatrix}$$

Can also be expressed as T(xf,yf).S(sx,sy).T(-xf,-yf) = S(xf, yf, sx, sy)

Transformations Properties

- Concatenation properties

  $A.B.C = (A.B).C = A.(B.C)$

  Matrix products can be evaluated from left to right or from right to left

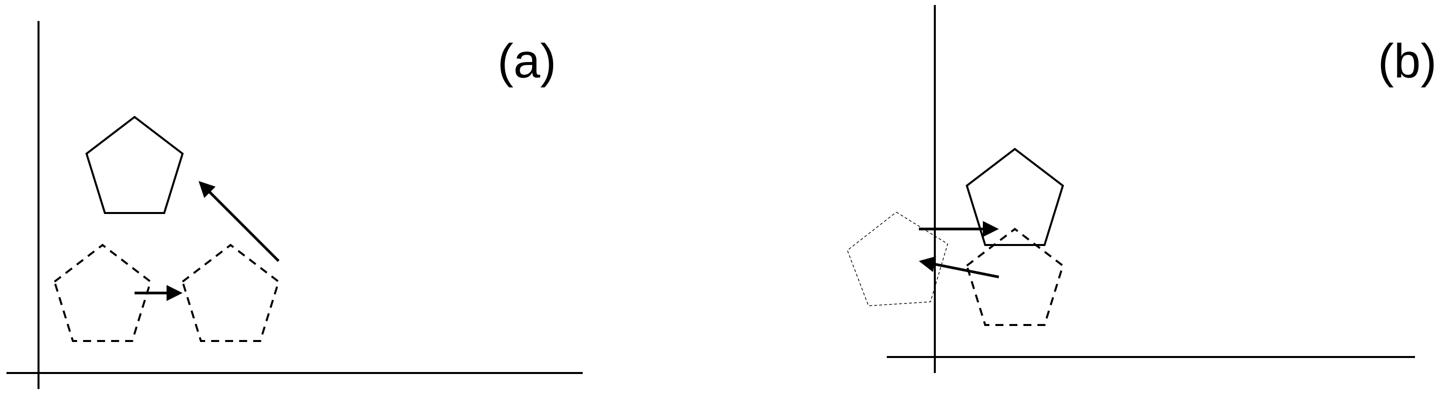
- However they are not commutative

  $A.B \neq B.A$

- Hence one must be careful in order that the composite transformation matrix is evaluated

Order of Transformations

- Reversing the order in which a sequence of transformations is performed may effect the transformed position of an object.



- In (a) object is first translated , then rotated
- In (b) the object is rotated first and then translated

General Composite transformations

- A general 2D transformation representing a combination of translations, rotations and scalings are expressed as,

$$\begin{bmatrix} X' \\ Y' \\ 1 \end{bmatrix} = \begin{bmatrix} rs_{xx} & rs_{xy} & trs_{x} \\ rs_{yx} & rs_{yy} & trs_{y} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

- rsij represent multiplicative rotation-scaling terms , trsx and trsy are the translational terms containing combinations of translations and scaling parameters

Composite transformations

- For example , if an object is to be scaled and rotated about its center coordinates (xc,yc) and then translated , the composite transformation matrix looks like
- $T(tx,ty) . R(xc,yc,\theta) . S(xc, yc, sx, sy)$

$$
= \begin{bmatrix} sx\cos\theta & -sy\sin\theta & xc(1-sx\cos\theta) + yc\ sy\sin\theta + tx \\ sx\sin\theta & sy\cos\theta & yc(1-sy\cos\theta) - xc\ sx\sin\theta + ty \\ 0 & 0 & 1 \end{bmatrix}
$$

Represents 9 multiplications and 6 additions

Composite Transformations

- The explicit calculation has only 4 multiplications and 4 additions

$$x' = x.rs_{xx} + y.r\,s_{xy} + tr\,s_x$$
$$y' = x.\,rs_{yx} + y.\,rs_{yy} + tr\,s_y$$

- The efficient implementation is to
  - Formulate transformation matrices
  - Concatenate transformation sequence
  - Calculate transformed coordinates using the explicit equation shown above

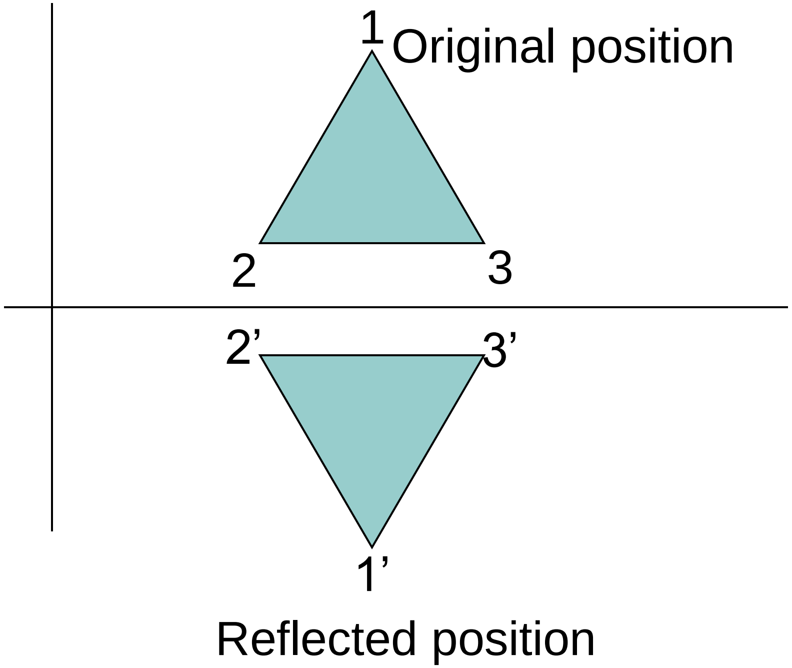
A general rigid body transformation matrix

- A general rigid body transformation involving only translations and rotations can be expressed in the form

$$\begin{bmatrix} r_{xx} & r_{xy} & tr_x \\ r_{yx} & r_{yy} & tr_y \\ 0 & 0 & 1 \end{bmatrix}$$

Other transformations

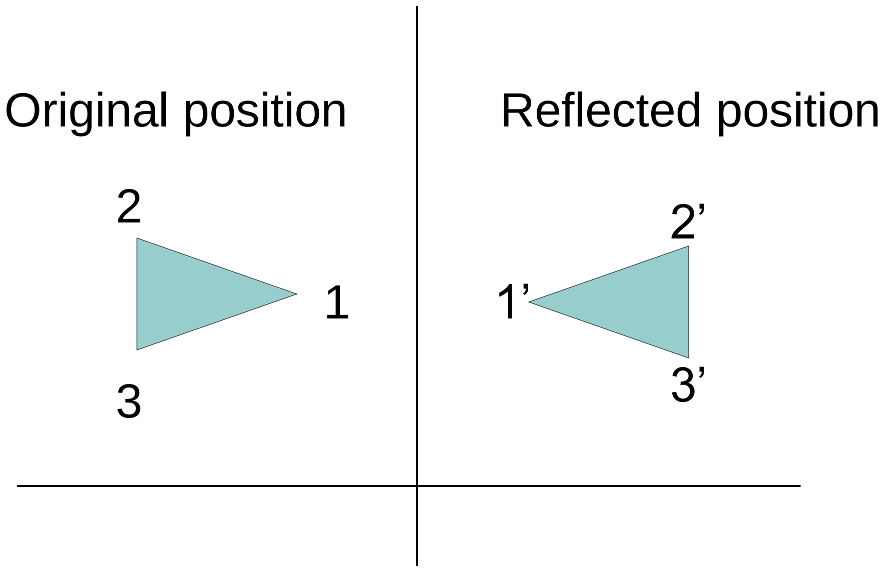
- Reflection is a transformation that produces a mirror image of an object. It is obtained by rotating the object by 180 deg about the reflection axis



Reflection about the line y=0, the axis , is accomplished with the transformation matrix

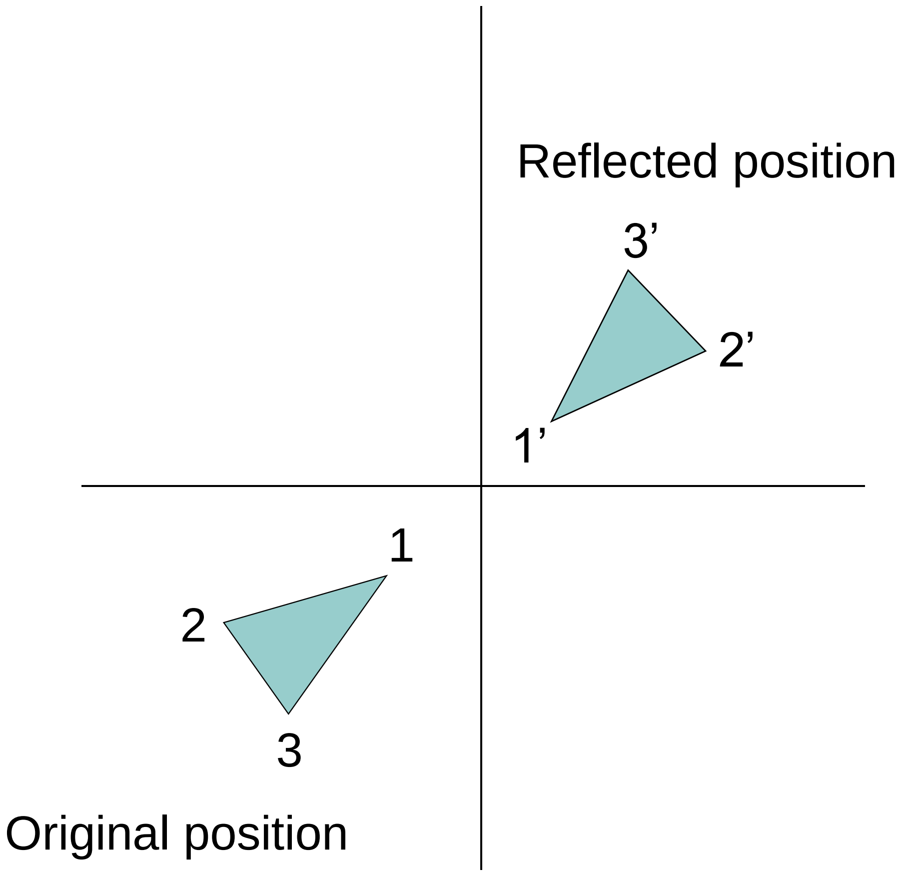
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Reflection



$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

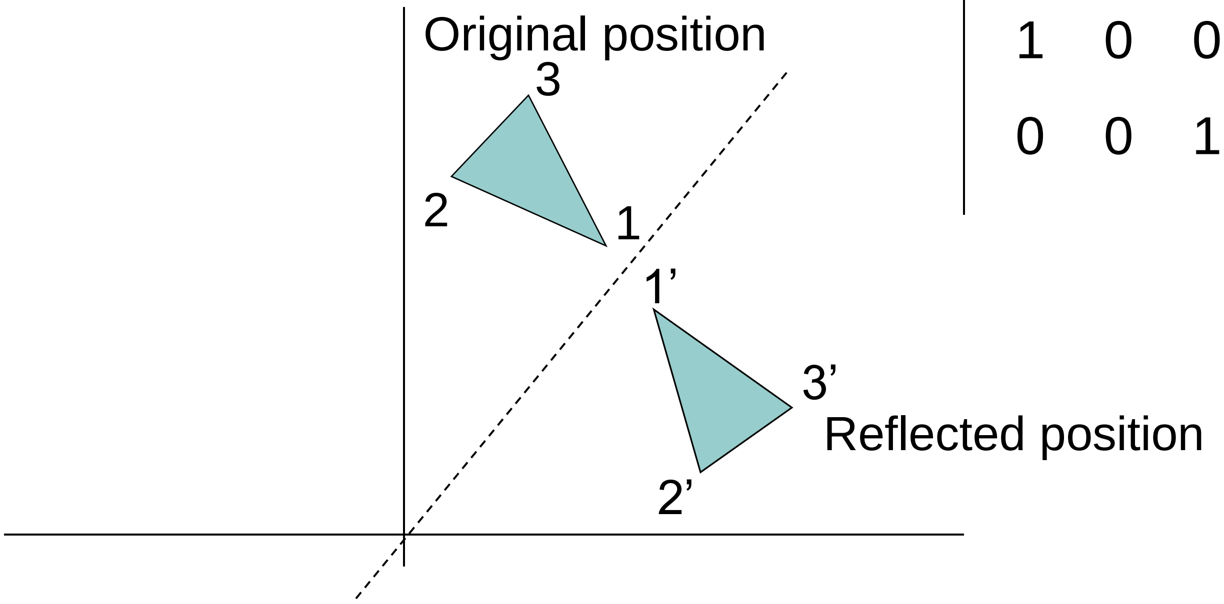
Reflection



$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Reflection of an object relative to an axis perpendicular to the xy plane and passing through the coordinate origin

Reflection of an object w.r.t the line y=x



$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Shear Transformations

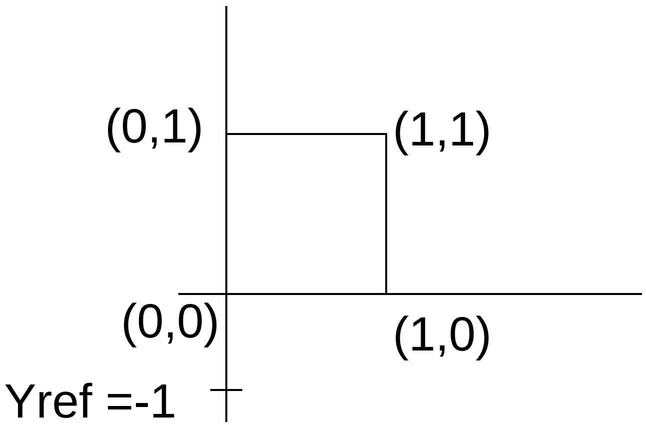
- Shear is a transformation that distorts the shape of an object such that the transformed shape appears as if the object were composed of internal layers that had been caused to slide over each other
- Two common shearing transformations are those that shift coordinate x values and those that shift y values
- An x direction shear relative to the x axis is produced with the transformation matrix

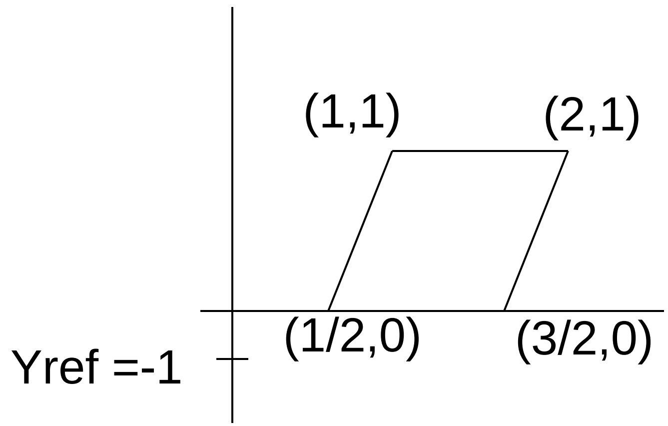
$$\begin{bmatrix} 1 & shx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which transforms coordinate positions as

$X' = x + shx.y$ $y' = y$

Shear tranformations





We can generate x-direction shears relative to other referance lines with

$$\begin{vmatrix} 1 & shx & -shx\ yref \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

With coordinate positions transformed as

X'= x + shx (y - yref)   y' = y

Shear transformation

A y-direction shear relative to the line x = xref is generated with the transformation matrix

$$\begin{vmatrix} 1 & 0 & 0 \\ Shy & 1 & -shy.xref \\ 0 & 0 & 1 \end{vmatrix}$$

This generates transformed coordinate positions

$X' = x \quad y' = shy\,(x - xref) + y$